Introduction to Magnetic Resonance Imaging

Loan Vo, John Stastny

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1 Objective

The objective of this laboratory is to familiarize the reader with the basic principle of magnetic resonance imaging. This includes signal generation, data reconstruction, and sources of noise and artifacts in MRI. In addition, an emphasis will be placed on signal processing concepts in MRI including the k-space, temporal and spatial under sampling, filtering, and Fourier transform relationships.

2 Introduction

Magnetic resonance imaging (MRI) is an extremely versatile imaging modality, with a broad range of clinical applications. Some of the most important applications of MRI are cardiac imaging, breast cancer imaging, brain imaging, fMRI, and spectroscopic imaging. MRI is based on the principle of spin quantum mechanics. However, the well known Bloch equations provide a macroscopic description of MRI which is considerably easier to analyze. In this lab, we will begin by looking at the basic principles behind MRI signal generation.

2.1 Signal Generation

In MRI, we are concerned with three magnetic fields. These three magnetic fields are the main magnetic field (commonly called the B0 field), the B1 field (also called the RF pulse), and the gradient field (responsible for encoding spatial information). Next, we will look at each of these fields, and their role in MRI.

2.1.1 The B0 field

In MRI the B0 field is a strong permanent magnetic field, typically 1 Tesla to 3 Tesla in strength. For some perspective, the earths magnetic field is around .5 gauss, where 1 gauss = 10^{-4} Tesla. The main magnetic field serves two roles in MRI. To understand these roles, we first look at the atomic basis for MRI.

We can think of individual atoms as magnetic dipoles, or spins. According to this description, the magnetic moment of a spin is given by:

\[ \mu = \gamma J \]  

where \( J \) is the angular momentum, and \( \gamma \) is the gyromagnetic ration, which for hydrogen is given by

\[ \gamma = 42.58 \text{MHz/T} \]

The magnitude of the magnetic moment is given by:

\[ |\mu| = \gamma \hbar \sqrt{I(I+1)} \]  

where \( I \) is the spin quantum number, and \( \hbar \) is planks constant over \( 2\pi \). In the case of spin \( \frac{1}{2} \) atoms, \( \mu \) is always \( \sqrt{3}/2 \). It is important to note that \( \mu \) is a vector quantity. In other words,
Figure 1: Spin angular momentum and associated projection onto z-axis

\[ J = \hat{x}J_x + \hat{y}J_y + \hat{z}J_z \]

The two possible orientations of \( \mu \) with respect to a fixed axis (taken to be the z-axis) are shown in figure 1.

The bulk magnetization is given by the ensemble over the entire sample:

\[ M = \sum_{n=1}^{N_s} \mu_n \] (3)

where \( N_s \) is the number of spins in the ensemble, and \( \mu_n \) is given by (1).

In general, an ensemble of spins will possess no bulk magnetization, since each spin has random angular momentum, so that the summation over a large number of systems will give zero as shown in figure 3. However, if we place an ensemble of spins into a strong magnetic field oriented in the z-direction, the z-component of \( \mu_n \) will cease to be random. Instead, the distribution of \( \mu_{n,z} \) will be given by the following expression;

\[ \frac{N_\uparrow}{N_\downarrow} = e^{\frac{\Delta E}{K T_s}} \] (4)

where \( \Delta E \) is the energy difference between the spin up and spin down states, \( T_s \) is the temperature in kelvin, and \( K \) is Boltzman’s constant.

The energy corresponding to spin up and spin down states is given by \( +\frac{1}{2}\gamma\hbar B_0 \) and \( -\frac{1}{2}\gamma\hbar B_0 \). The energy diagram is shown in figure 2. This effect is commonly known as Zeeman splitting. In this figure, the frequency, rather than the energy is given. Note, that the energy difference, \( \Delta E \) is given by:

\[ \Delta E = \gamma\hbar B_0 \] (5)

which corresponds to a frequency, \( 2\pi\gamma B_0 = \omega_0 \).

By first order Taylor series approximation to (4), we may write:

\[ N_\uparrow - N_\downarrow \approx N_s \frac{\gamma\hbar B_0}{2KT_s} \] (6)

Equation (6) indicates there is a very small excess number of spin ups.

Using (3) and (4), we find the total z bulk magnetization is given by:
As an example, we consider the value for (7) for some typical values found in MRI. Take the following parameters:

\[ T_s = 300K \]
\[ B_0 = 1\text{Tesla} \]

Then we get the following:

\[ \frac{N_\uparrow - N_\downarrow}{N_s} = \frac{\gamma \hbar B_0}{2KT_s} = 3 \times 10^{-6} \]  

It is for this reason that MRI is known as a low sensitivity imaging modality. That is, a very small difference in the number of spin ups to spin downs produces all of the signal in MRI.

Remember that the total bulk x and y magnetization is still zero, since the x and y components of the angular momentum are random, and hence sum to zero.
In addition to creating the Zeeman splitting effect, the $B_0$ field causes the spins to precess about the $z$-axis at a frequency known as the larmor frequency, given by:

$$\omega_0 = \gamma B_0$$  \hspace{1cm} (9)

This frequency also corresponds to the energy difference between the spin up and spin down states. So in summary, the $B_0$ field is responsible for the Zeeman splitting effect, and the precession of spins.

Next, we look at the $B_1$ field or RF field. From this point on, we consider the bulk magnetization model, where the total $z$-magnetization in the presence of a magnetic field $B_0$ is given by (7).

### 2.1.2 The $B_1$ or RF field

The $B_1$ field is an RF (radio frequency) pulse transmitted at a frequency corresponding to the Larmor or resonance frequency, $\omega_0$. The manipulation of the bulk magnetization begins with the application of the RF pulses. To understand how RF pulses work, we consider again the energy diagram in figure (1). According to planks formula, a photon of frequency $\omega$, has energy,

$$E = \hbar \omega$$

Therefore, hitting a spin with a photon with frequency $\omega_0 = \gamma B_0$, will cause that spin to change states. In other words, when a spin with energy $+\frac{1}{2}\omega_0$ is hit with a photon of frequency $\omega$, it will jump to the spin down state, with an energy $-\frac{1}{2}\omega_0$. However, this is an unstable state. This spin will eventually return to a spin up state, and in the process will emit a photon of frequency $\omega_0$.

This is the basic principle behind MRI, and the application of RF pulses. The RF pulse causes spins to change to the opposite state. When these spins return to their original state, they emit photons (RF energy) at the same frequency, $\omega_0$.

We begin by considering the equilibrium magnetization, which we represent as a vector as shown in figure 4. Initially, all magnetization is along the $z$-direction. The RF pulse can be applied in any direction ($x$, $y$, -$x$, -$y$). The effect of the RF pulse on the magnetization vector, $M$ is to rotate the vector about the axis of RF application by an angle

$$\alpha = \gamma B_1 \tau$$

where $B_1$ is the strength of the RF pulse, and $\tau$ is the length of the RF pulse.

For example, consider the effect of an $\alpha_x$ pulse on the bulk magnetization vector,

$$M = \begin{pmatrix} M_x^0 \\ 0 \\ 0 \end{pmatrix}$$

Immediately after the pulse, the magnetization components are:

$$\begin{align*}
M_x'(0_+) &= 0 \\
M_y'(0_+) &= M_z^0 \sin \alpha \\
M_z'(0_+) &= M_z^0 \cos \alpha
\end{align*}$$  \hspace{1cm} (10)
In general, the effect of an RF pulse on a general magnetization vector can be described by a rotational matrix:

$$R_{x'}(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{pmatrix}$$ (11)

$$R_{y'}(\alpha) = \begin{pmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{pmatrix}$$ (12)

$$R_{z'}(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ (13)

After the application of an RF pulse, the transverse magnetization will undergo relaxation, where by the magnetization returns to its equilibrium value. There are two types of relaxation; longitudinal relaxation and transverse relaxation. Transverse relaxation describes the decay of the transverse magnetization, $M_{x'y'}$. Longitudinal relaxation refers to the return of the longitudinal magnetization $M_z$ to its equilibrium value, $M_0^z$.

$$\begin{cases} M_{x'y'}(t) = M_{x'y'}(0) e^{-\frac{t}{T_2}} \\ M_z(t) = M_0^z(1 - e^{-\frac{t}{T_1}}) + M_z(0) e^{-\frac{t}{T_1}} \end{cases}$$ (14)

### 2.1.3 Free Induction Decays and Spin Echos

A free induction decay, or FID is the result of a single RF pulse on a spin system. Free refers to the fact that the signal results from the free precession of the bulk magnetization about the $B_0$ field. Induction indicates the signal was generated based on Faraday’s law of induction. Decay refers to the decrease characteristic in signal intensity with time, which is a result of the relaxation effects described above.

An FID signal resulting from an $\alpha$ pulse is given by:
\[ S(t) = \sin \alpha \int_{-\infty}^{\infty} \rho(\omega)e^{-\frac{t}{T_2}}e^{-i\omega t}d\omega \]  

(15)

where \( \rho(\omega) \) is the spectral density function, which characterizes the spin system. For example, the FID from a system with a single spectral component is given by:

\[ S(t) = M_0^0 \sin \alpha e^{-\frac{t}{T_2}}e^{-i\omega_0 t} \]  

(16)

Spin echoes are the result of the dephasing and rephasing of spins as a result of the application of two or more RF pulses. In the simplest case, we apply the pulse sequence:

\[ 90 - \tau - 180 \]

After the 90 degree pulse, the bulk magnetization vectors lie along the \( y' \) axis. At this point, we consider two isochromats, at slightly different resonance frequencies, \( \omega_s \) and \( \omega_f \), representing a slow and fast isochromat respectively. During the delay, \( \tau \), these two isochromats dephase. When the 180 degree pulse is applied, both of the bulk magnetization vectors flip over the \( y' \) axis. Finally, after another delay, \( \tau \), the two isochromats rephase, creating a spin echo. The pulse sequence and the associated spin echo signal are shown in ??

### 2.2 Gradient Fields: Encoding Spatial Information

After a signal has been activated by an RF pulse, spatial information can be encoded during the free precession period. There are two main ways to encode spatial information in MRI; frequency encoding and phase encoding.

#### 2.2.1 Frequency Encoding

Frequency encoding makes the oscillation frequency of an activated MRI signal linearly dependent on spatial position. We first consider a one dimensional object, with an associated spin density \( \rho(x) \). If we assume the object is exposed to a uniform \( B_0 \) field, then under the influence of a linear gradient, \( G(x) \), the Larmor frequency at a position \( x \) is given by:

\[ \omega(x) = \omega_0 + \gamma G x \]  

(17)

Neglecting the transverse relaxation effect, the associated FID for a small area is:

\[ dS(x, t) = \rho(x)dx e^{-i\gamma(B_0+G x)t}dx \]  

(18)

The received signal from the entire object under frequency encoding gradient is given by:

\[ S(t) = \int_{-\infty}^{\infty} \rho(x)e^{-i\gamma(B_0+G x)t}dx = \left[ \int_{-\infty}^{\infty} \rho(x)e^{-i\gamma G x t}dx \right] e^{-i\omega_0 t} \]  

(19)

After demodulation (removal of carrier signal \( e^{-i\omega_0 t} \)), we get:

\[ \int_{-\infty}^{\infty} \rho(x)e^{-i\gamma G x t}dx \]  

(20)
In the more general case, where our phase encoding gradient is given by

\[ G_{fe} = (G_x, G_y, G_z) \]

The signal expression in equation (20) is replaced by the expression:

\[ S(t) = \int_{\text{object}} \rho(\mathbf{r}) e^{-i\gamma G_{fe} \cdot \mathbf{r}} d\mathbf{r} \quad (21) \]

### 2.2.2 Phase Encoding

The principle behind phase encoding is very similar to that of frequency encoding. We begin once again with a simple 1-D case. If we apply a gradient \( G_x \) for a short period of time \( T_{pe} \), then our local signal is given by:

\[ dS(x, t) = \begin{cases} 
\rho(x) e^{-i\gamma(B_0 + G_xx)t} & 0 \leq t \leq T_{pe} \\
\rho(x) e^{-i\gamma G_xx T_{pe}} e^{-i\omega_0 T_{pe}} & T_{pe} \leq t 
\end{cases} \quad (22) \]

After the phase encoding gradient is shut off, all of the spin systems will again precess at the same frequency \( \omega_0 \). However, each will have a different phase, given by:

\[ \phi(x) = -\gamma G_xx T_{pe} \quad (23) \]

### 2.3 The K-space Interpretation

In this section, we draw the connection between spatial encoding (phase encoding and frequency encoding), and the Fourier Transform. This connection provides us the means to analyze complex pulse sequences using the k-space notation.

#### 2.3.1 Frequency Encoded Signals

We first consider the frequency encoded signal, given by (20). Making a simple variable substitution, we obtain the following Fourier transform relationship.

\[ S(k_x) = \int_{-\infty}^{\infty} \rho(x) e^{-i2\pi k_x x} dx \quad (24) \]

where \( k_x \) is given by:

\[ k_x = \begin{cases} 
2\pi\gamma G_x t & \text{FID signals} \\
2\pi\gamma G_x (t - T_E) & \text{echo signals} 
\end{cases} \quad (25) \]

This expression can be easily extended to the case when we have multiple frequency encoding gradient:

\[ k = \begin{cases} 
2\pi\gamma G_{fe} t & \text{FID signals} \\
2\pi\gamma G_{fe} (t - T_E) & \text{echo signals} 
\end{cases} \quad (26) \]

The corresponding k-space signal according to is,
\[ S(k) = \int_{\text{object}} \rho(r)e^{-i2\pi\mathbf{k} \cdot \mathbf{r}} \, d\mathbf{r} \quad (27) \]

It is important to note that although the k-space signal is a multidimensional Fourier transform, the signal \( S(k) \), is available only at a limited set of discrete points in k-space. These k-space points define the sampling trajectory of k-space.

For example, in the two dimensional case where we have an FID which is frequency encoded, we get the following k-space sampling trajectory.

\[
\begin{aligned}
{k_x} &= 2\pi\gamma G_x t \\
{k_y} &= 2\pi\gamma G_y t
\end{aligned}
\quad (28)
\]
or

\[
\begin{aligned}
{k_x} &= k \cos \phi \\
{k_y} &= k \sin \phi
\end{aligned}
\quad (29)
\]

where

\[ k = 2\pi\gamma G_f et = 2\pi\gamma t \sqrt{G_x^2 + G_y^2} \quad (30) \]

and

\[ \phi = tan^{-1} \left( \frac{G_y}{G_x} \right) \quad (31) \]

Equation 29 describes a straight line through the origin of k-space as shown in figure ???. These same conclusions can be easily extended to 3 dimensions. In this case, we have:

\[
\begin{aligned}
{k_x} &= k \sin \theta \cos \phi \\
{k_y} &= k \sin \theta \sin \phi \\
{k_z} &= k \cos \theta
\end{aligned}
\quad (32)
\]

where

\[ k = 2\pi\gamma G_f et = 2\pi\gamma t \sqrt{G_x^2 + G_y^2 + G_z^2} \quad (33) \]

\[ \theta = tan^{-1} \left( \frac{\sqrt{G_x^2 + G_y^2}}{G_z} \right) \quad (34) \]

and

\[ \phi = tan^{-1} \left( \frac{G_y}{G_x} \right) \quad (35) \]

In the most general case, where \( G_{fe} \) is a function of time, the mapping between \( G_{fe}(t) \) and \( k \) is given by;

\[ \mathbf{k}(t) = 2\pi\gamma \int_0^t G_{fe}(\tau) d\tau \quad (36) \]

One very important point to address is the effect of a 180 degree RF pulse on the k-space trajectory. The following example will address this issue.
Recall that the effect of a 180 degree pulse on the transverse magnetization is given by:

\[ M_{x'y'} \xrightarrow{\pi/2} M_{x'y'}^* e^{-i2\phi} \]  

(37)

which can also be expressed as:

\[ M_{x'y'} = \rho(r) dr e^{i2\pi kr} \]  

(38)

then the postpulse transverse magnetization is given by;

\[ M_{x'y'} = \rho(r) dr e^{i(2\pi kr - 2\phi)} \]  

(39)

It is clear from equations (38) and (39) that

\[ k \xrightarrow{\pi/2} -k \]  

(40)

2.3.2 Phase Encoding Gradients

Next, we consider the effect of the phase encoding gradient in terms of the Fourier Transform. We can express the phase encoded signal in 22 as

\[ S(k) = \int_{\text{object}} \rho(r) e^{-i2\pi kr} dr \]  

(41)

if we drop the carrier signal, \( e^{-i\omega_0 t} \), and make the variable substitution:

\[ k = \frac{\gamma}{2\pi} G_{pe} T_{pe} \]  

(42)

It is important to note that, unlike in the case of the frequency encode gradient, we are not taking samples of k-space during a phase encode. Rather, we simply travel in k space according to equation (42). In the general case, where we do not have a rectangular phase encode gradient, \( k \) is given by:

\[ S(k) = \frac{\gamma}{2\pi} \int_0^{T_{pe}} G_{pe}(\tau) d\tau \]  

(43)

2.4 Basic Imaging Pulse Sequence

In this section, we look at the most basic imaging sequence, and consider the associated k-space sampling trajectory. The pulse sequence to be considered is shown in figure 5.

To understand how the k-space is sampled in this scheme, we consider the \( n_{th} \) excitation.

\[ \begin{align*}
  k_x &= \frac{\gamma}{2\pi} G_x (t - t_0) & t_0 < t < T_{acq}/2 + t_0 \\
  k_y &= \frac{\gamma}{2\pi} n \Delta G_y (t - t_0)
\end{align*} \]  

(44)

Which represents a radial line from the origin to point A, defined by:

\[ k_A = \left( \frac{\gamma}{2\pi} G_x T_{pe}, \frac{\gamma}{2\pi} n \Delta G_y T_{pe} \right) \]  

(45)

The effect of the 180 degree pulse is to flip the k space point to B:
Figure 5: Basic Imaging pulse sequence.

Figure 6: The k-space trajectory for 1 repetition of the basic imaging pulse sequence.

\[ k_B = -k_A \]  \hspace{1cm} (46)

During the data acquisition interval, the k-space sampling points are given by:

\[ \begin{align*}
  k_x &= \gamma G_x (t - T_E) \quad |t - T_E| < T_{acq}/2 \\
  k_y &= \gamma n \Delta G_y T_{pe}
\end{align*} \]  \hspace{1cm} (47)

The corresponding k-space trajectory of this one excitation is shown in figure 6.

This represents a single horizontal line in k-space. In this imaging scheme, we repeat this pulse sequence N times, changing the phase encode gradient \( G_y \) each time. In this way, we collect N lines in k-space. With this k-space data, we can recover the underlying image function, \( \rho(x, y) \). It is important to note here that the time required for the entire imaging experiment is roughly proportional to the number of phase encodes. That is, application of the pulse sequence shown in 5 takes a time \( T_R \), then the total time for this experiment is:
The number of frequency encode points does not contribute very much to the acquisition time, since it requires only more samples (higher sampling rate).

3 Fourier Reconstruction, Noise and Image Artifacts

3.1 Fourier Reconstruction

For a rectilinear sampling trajectory, the collected data are related to the reconstructed image $I(x)$ by:

$$
\Delta k \sum_{n=-\infty}^{\infty} S[n] e^{j2\pi kn} = \frac{1}{\Delta k} \sum_{n=-\infty}^{\infty} I(x - \frac{n}{\Delta k})
$$

(49)

In the above equation, the left-hand side has the form of a Fourier Series with a fundamental frequency $\Delta k$. $S[n]$ represents the coefficient of the $n^{th}$ harmonic component in this Fourier Series. On the right hand side, we have periodic replicas of the image $I(x)$.

If we assume that we have infinite samples of $S[n]$ e.g. $-\infty < n < \infty$, then the desired image is simply a filtered version (choosing only one replica) of (8), provided we choose $\Delta k$ to satisfy the Nyquist sampling criterion. The Nyquist sampling criterion in this case is given by:

$$
\Delta k < \frac{1}{W_x}
$$

(50)

where $W_x$ is the field of view, ie $I(x) = 0 \forall |x| > W_x/2$. In this case, the desired image can be described as;
Figure 8: Replication of the reconstructed image (without aliasing artifact)

Figure 9: Replication of the reconstructed image with aliasing artifact in one direction
\[ I(x) = \Delta k \sum_{-\infty}^{\infty} S[n] e^{i2\pi k_n x} |x| < \frac{1}{2\Delta k} \] (51)

However, in practice it is not possible to collect an infinite number of samples. Rather, we collect \( N \) samples, such that \( S[n] - N/2 \leq n < N/2 \). Therefore, the reconstructed image is not unique. We call the set of all possible reconstructions the set of feasible solutions. Although many solutions exist, we often choose one based on some additional condition. Most commonly, we choose the solution with the minimum norm, called the minimum norm reconstruction:

\[ I(x) = \Delta k \sum_{-N/2}^{N/2-1} S[n] e^{i2\pi k_n x} |x| < \frac{1}{2\Delta k} \] (52)

Later, we will show that any image reconstructed from a finite number of samples will contain Gibb’s ringing artifacts.

In addition to acquiring our data in the discrete domain, we usually process our image in the discrete domain, where \( I[m] = I(m\Delta x) \). In this case, we choose the maximum Fourier pixel size \( \Delta x = \frac{1}{N\Delta k} \) so that Nyquist criterion can be satisfying when sampling the continuous image into digital image domain. Thus, we have the following DFT (Discrete Fourier Transform) relationship between the discrete collected data and the digital reconstructed image:

\[ I[m] = \Delta k \sum_{-N/2}^{N/2-1} S[n] e^{i2\pi nm/N} - N/2 \leq m < N/2 \] (53)

Note that the reconstructed image can be obtained without knowledge of \( \Delta k \). Therefore, we usually assume \( \Delta k = 1 \) (and hence \( \Delta x = \frac{1}{N} \))

\[ I[m] = \sum_{-N/2}^{N/2-1} S[n] e^{i2\pi nm/N} - N/2 \leq m < N/2 \] (54)

Like the Fourier Transform for continuous signals, the DFT transforms a discrete time domain signal into its discrete frequency representation. The main advantage of working in the discrete domain is the ability to easily manipulate signals and images on computers, which inherently work with discrete data. The Fast Fourier Transform (FFT) is an extremely fast implementation of the discrete Fourier Transform, which is often used in processing discrete signals.

To increase the digital resolution of the image, the discrete frequency domain data is often zero-padded. This zero-padding in the frequency domain has the effect of interpolation in the space domain. However, it is important to realize that the Fourier resolution \( \Delta x \) cannot be changed by zero-padding. Rather, \( \Delta x \) depends only on the total number of data points collected \( N \) (with \( \Delta k = 1 \)).
3.2 Noise

In MRI, the acquired k-space data are contaminated by noise. To analyze the noise characteristics of the acquired signal, and how this noise effects the reconstructed image, we consider the following. We assume that the noise at all k-space locations, is independent and identically distributed (iid) gaussian, with zero mean and variance $\sigma_d^2$. We denote this noise as $\xi_d[m]$. Furthermore, we denote the noise corrupted k-space signal and noise corrupted image as $\hat{S}, \hat{I}$, respectively. The clean k-space signal and clean image are represented as $S$ and $I$, respectively.

We assume the following additive model for the noise corrupted k-space signal $\hat{S}(k)$:

$$\hat{S}(k) = S(k) + \xi_d(k)$$  \hspace{1cm} (55)

Taking the 2-D Fourier transform of this noise corrupted data results in the noisy image:

$$\hat{I}(x) = I(x) + \xi_I(x)$$  \hspace{1cm} (56)

3.2.1 Noise in direct FFT Reconstruction

Next, we consider the case of applying a discrete Fourier Transform to the discrete data in the presence of noise.

In this case, the image noise is related to the k-space noise by:

$$\xi[m] = \frac{1}{N} \sum_{-N/2}^{N/2-1} \xi_d[n] e^{j2\pi mn/N} - N/2 \leq m < N/2$$  \hspace{1cm} (57)

Using this expression, we find the first order statistics of the image noise:

$$E\{\xi_I[m]\} = 0$$  \hspace{1cm} (58)

$$\sigma_I^2 = \frac{1}{N} \sigma_d^2$$  \hspace{1cm} (59)

$$E\{\xi_I[m] \xi_I^*[n]\} = 0 \quad m \neq n$$  \hspace{1cm} (60)

Using (60), it is easy to prove that the signal to noise ratio (SNR) at each pixel location is:

$$SNR_{\text{pixel}} = \frac{I_{\text{avg}}}{\sigma_I}$$  \hspace{1cm} (61)

$$= \frac{1}{N} \sum_{-N/2}^{N/2-1} I[m] I^*[m]$$  \hspace{1cm} (62)

$$= \sqrt{\sum_{n=-N/2}^{N/2-1} |S[n]|^2}$$  \hspace{1cm} (63)

It is clear from this expression that the SNR decreases as the total number of encodings increases. This trade off between spatial resolution, (primarily in the frequency encode
direction), and SNR is important to keep in mind. Although increasing the number of frequency encodings may not greatly increase the acquisition time, it will degrade the SNR.

In the image domain, the noise still has uncorrelated variance from pixel to pixel. Therefore, SNR can be improved by taking the average value of several adjacent pixels in an image. Once again, we trade off spatial resolution for SNR. We denote \( \hat{I_1}[m] \) as the noise-corrupted FFT image and \( \hat{I_2}[m] \) as:

\[
\hat{I_2}[m] = \frac{1}{P} \sum_{p=0}^{P-1} \hat{I_1}(m + p)
\]

where \( P \) is the number of adjacent pixels averaged. Then it is clear that the SNR of \( \hat{I_2}[m] \) is improved by a factor of \( \sqrt{P} \) as compare with SNR of \( \hat{I_1}[m] \)

### 3.2.2 Noise in zero-padded FFT Reconstruction

In this section, we explore the effect of zero-padding on the noise characteristics. We consider zero-padding our k-space signal from \( N \) points to \( M \) point, to produce an image of length \( M \). In this case, our new image noise is given by:

\[
\xi[m] = \frac{1}{N} \sum_{-N/2}^{N/2-1} \xi_d[n]e^{j2\pi nm/N} - M/2 \leq m < M/2
\]

Therefore, we also have:

\[
E\{\xi_I[m]\} = 0
\]

\[
\sigma^2_I = \frac{1}{N} \sigma^2_d
\]

\[
E\{\xi_I[m]x^*_I[n]\} = \frac{1}{N^2} \sigma^2_d \frac{\sin[\pi(m-n)/M]}{\sin[\pi(m-n)/M]} e^{-i\pi(m-n)/N} \quad m \neq n
\]

Zero-padding does not change the image noise mean or variance. Consequently, the SNR in the zero-padded image is identical to that of the non zero-padded image. At this point, the observent reader may ask why we do not simply average the extra adjacent pixels to increase SNR by a factor of \( \sqrt{P} \). In this case, the noise at adjacent pixels is not independent, and so averaging adjacent pixels will not improve SNR.

### 3.3 Image Artifacts

#### 3.3.1 Gibbs Ringing Artifact

In this section, we explore the phenomena of Gibbs ringing in MRI. In MRI, we must truncate our k-space data to collect a finite number of k-space samples. As a result, the high frequency components of the data are discarded. The resulting image is actually the result of a convolution between the ideal image and the point spread function:
\[ h(x) = \Delta k \sum_{n=-N/2}^{N/2-1} e^{-i\pi \Delta k x} \]  
\[ = \Delta k \frac{\sin(\pi N \Delta k x)}{\sin(\pi \Delta k x)} e^{-i\pi \Delta k x} \]  
\[ (71) \]

This artifact is commonly referred to as the Gibbs Ringing Artifact. The Gibbs phenomenon causes spurious ringing around sharp edges (Also note that, the maximum undershoot or overshoot of the spurious ringing is about 9% of the intensity discontinuity and it does NOT depend on the number of data points used in the reconstruction).

In a 2D reconstructed image, Gibbs ringing can appear in both frequency and phase encoding directions. However, the Gibbs ringing artifact is more pronounced in the phase encoding direction. This is because the number of phase encodes we can collect is heavily constrained by the desired imaging time.

Although the Gibbs ringing artifact can be reduced by acquiring more high frequency k-space points, in practice it is not always possible due to time requirements. However, as we have discussed, it is relatively easy to increase the number of frequency encode points, and therefore reduce the Gibbs ringing artifact in this direction. Another commonly used approach in reducing the Gibbs ringing artifact is known as windowing. In windowing, a window function is used to suppress the oscillations of the original point spread function:

\[ \hat{I}(x) = \Delta k \sum_{n=-N/2}^{N/2-1} S(n \Delta k) w_n e^{j2\pi k_n x} \]  
\[ (72) \]

where \( w_n \) is the window function, which clearly acts as a filter. The reduction in Gibbs ringing artifact by a window function can be understood by looking at the point spread function with windowing, \( h(x) = \Delta k \sum_{n=-N/2}^{N/2-1} w_n e^{-i\pi \Delta k x} \). A proper choice of window functions can lead to a smoother behavior in the overall point spread function.

Although applying a window function can reduce the Gibbs ringing artifact, a trade-off does exist. This trade off is a reduction in spatial resolution (blurring). Spatial resolution is reduced due to an increase in the effective width of the point spread function after windowing.

In addition to windowing, several other techniques exist for reducing Gibbs ringing. For example, the Generalized Series reconstruction (known as RIGR or TRIGR) algorithm takes advantage of known information regarding the data to reconstruct high spatial resolution images from a limited number of k-space points.

### 3.3.2 Aliasing Artifact

As mentioned in (49), the Fourier series reconstruction of the collected data \( S(k) \) represents many replicas or the desired image:

\[ \hat{I}(x) = \sum_{n=-\infty}^{\infty} I(x - \frac{n}{\Delta k}) \]  
\[ (73) \]
Figure 10: Gibbs ringing artifact. Top-left image is the original phantom image. Top-right, bottom-left and bottom-right images have Gibbs ringing in phase-direction only, frequency-direction only and both of the frequency and phase direction.

Violating the Nyquist condition (under-sampling in k-space) will cause these replicas to wrap-around and overlap each other. This phenomena is commonly called aliasing. In general it is difficult to fix aliasing in post-processing. Therefore, it is important to satisfy the Nyquist criterion, or to use an anti-aliasing filter to limit the signal bandwidth.

3.3.3 Motion Artifact

In many practical applications, the object being imaged is not stationary. This leads to a so called motion artifacts. Depending on the acquisition scheme as well as the type of motion, artifacts appear in the reconstructed image in a variety of forms. Blurring and ghosting are the two most common motion artifacts. In this section, we are not going to discuss about techniques to overcome motion effects; however brief description how motion artifacts are introduced and appear in reconstructed images will be provided.

Since there are big differences between phase and frequency encodings, object motion effects are not the same in these two directions. The main difference is that motion effects along the phase direction seem to introduce more serious artifacts in reconstructed images because the time between two phase encodings is much larger than the time between two frequency encodings. However, since there is no phase accumulation between phase encodings, analyzing motion effect along the phase encoding is easier. Let now begin to investigate motion effect along the frequency encoding direction.

Motion Effects along the frequency-encoding direction (readout direction)
Figure 11: Aliasing artifact. Top-left image has no aliasing artifact. Top-right, bottom-left and bottom-right image have aliasing artifact with the reduced encoding factor equal to 2, 4 and 8.

Let assume that object motion has constant velocity $\vec{v} = v_x \vec{i} + v_y \vec{j}$. Also denote $I(x,y,0)$ as the object function at time 0 and $I(x,y,t)$ as the object at time instant t. We have:

$$I(x,y,t) = I(x(t), y(t), 0)$$  \hspace{1cm} (74)

where

$$x(t) = x + v_x t$$  
$$y(t) = y + v_y t$$  \hspace{1cm} (75)

As we know that readout gradient generates a phase component:

$$\phi = \gamma \int_0^t \vec{G}(\tau) \cdot \vec{r}(\tau) d\tau$$  \hspace{1cm} (76)

Assume that $\vec{G}(t) = G_x(t) \vec{i}$, then $\phi$ in equation (eqn:phiReadOut) can be expressed as:

$$\phi = \gamma \int_0^t G_x(\tau) (x + v_x \tau) d\tau = \gamma x \int_0^t G_x(\tau) d\tau + \gamma v_x \int_0^t \tau G_x(\tau) d\tau$$  \hspace{1cm} (77)

The first term in equation (77) is useful spatial encoding component. On the other hand, the second term leads to undesirable phase shifts - which are introduced by the motion velocity. Denote this motion-dependent phase shift as $\Delta \phi$, we have:
\[ \Delta \phi = \gamma v_x \int_0^t \tau G_x(\tau) d\tau \]  

(78)

If the readout gradient \( \vec{G}(t) \) has the following form:

\[ \vec{G} = \begin{cases} 
-G_x & 0 \leq t \leq \frac{T_E}{2} \\
0 & \frac{T_E}{2} \leq t \leq 3\frac{T_E}{2} 
\end{cases} \]

then the corresponding undesirable phase shift during the corresponding period is as following:

\[ \Delta \phi = \begin{cases} 
-\frac{1}{2} \gamma v_x G_x(t^2 - 2\frac{T_E}{2}) & 0 \leq t \leq \frac{T_E}{2} \\
\frac{1}{2} \gamma v_x G_x(t - 2\frac{T_E}{2}) & \frac{T_E}{2} \leq t \leq 3\frac{T_E}{2} 
\end{cases} \]

From equation 3.3.3, it is obvious that the peak of the echo signal is no longer at \( t = T_{acq} \).

In short, motion effects along the readout direction can be expressed in the following point spread function \( h(x) \) (see [1] for detailed derivations).

\[ h(x) \approx \frac{(1-i) \sqrt{\pi}}{\sqrt{\gamma G_x v_x}} e^{-\gamma G_x v_x (T_E)^2} e^{\frac{i\pi (x-x_s)^2}{v_x}} \]  

(79)

where \( x_s = v_x T_E \) - the distance object has moved in duration \( T_E \).

In equation (79), the term \( e^{-\gamma G_x v_x (T_E)^2} \) imposes phase shifts onto reconstructed images. If \( v_x \) is constants, this phase shift component can be ignore. But if \( v_x \) changes within a voxel, signal intensity will vary within the voxel. In general, along the direction in with \( v_x \) varies, dephasing will occur and create artifact in that direction (this artifact is analogous to the artifact introduced by motion in the phase encoding direction). The second term in the equation (79) leads to image blurring in reconstruction.

**Motion Effects along the phase-encoding direction**

Let assume that a constant phase encoding gradient \( G_n \) is used along the x-direction, starting at \( t_n \) and ending at \( t_n + \tau \) for the \( n^{th} \) phase encoding, we have:

\[ \phi_n = \int_{t_n}^{t_n+\tau} G_n(x + v_x t) dt = \gamma x \tau + \gamma G_n v_x (t_n + \tau / 2) = 2\pi k_n x + 2\pi k_n v_x (t_n + \tau / 2) \]

where \( k_n = \gamma / 2\pi G_n \tau \)

Hence, the undesirable phase shift along the phase encoding direction is

\[ \Delta \phi = 2\pi k_n v_x (t_n + \tau / 2) = 2\pi k_n v_x t_n + 2\pi k_n v_x \tau / 2 \]  

(80)

Since linear phase shift only introduces spatial displacement in the reconstruction, we won’t consider more the effect of the linear phase term \( 2\pi k_n v_x \tau / 2 \). Let rewrite \( \Delta \phi \) as following:

\[ \Delta \phi = 2\pi k_n v_x t_n \]  

(81)

Denote \( I(x, 0) \) (or \( I_0 \)) and \( I(x, t_n) \) are snapshot images at time 0 and time \( t_n \). Denote \( S_0(k_n) \) and \( S(k_n) \) as the collected signal at the \( n^{th} \) phase encodings of the object at time
0 ($I_0$) and time $t_n$ ($I(x, t_n)$) correspondingly. Then, the phase shift component in equation (81) imposes the following relationship between $S(k_n)$ and $S_0(k_n)$:

$$S(k_n) = \int_{-\infty}^{\infty} I(x, t_n) e^{-i2\pi k_n x} dx$$

$$= e^{-i2\pi k_n v_x t_n} \int_{-\infty}^{\infty} I_0 e^{-i2\pi k_n x} dx$$

$$= e^{-i2\pi k_n v_x t_n} S_0(k_n)$$

More general, we have:

$$S(k, t) = \int_{-\infty}^{\infty} I(x, t)e^{-i2\pi k x} dx \quad (82)$$

The above equation can be interpreted as simply as the fact that there is a Fourier transform relationship between object and collected data at any instantaneous time $t$. It also means that if we want to reconstructed image having no motion artifact we need to acquire the whole k-space data instantaneous.

**References**